

Binder cumulants of an urn model and Ising model above critical dimension

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Solving numerically master equation for a recently introduced urn model, we show that the fourth- and sixth-order cumulants remain constant along an exactly located line of critical points. Obtained values are in very good agreement with values predicted by Brézin and Zinn-Justin for the Ising model above the critical dimension. At the tricritical point cumulants acquire values which also agree with a suitably extended Brézin and Zinn-Justin approach.

The concept of universality and scale invariance plays a fundamental role in the theory of critical phenomena [1]. It is well known that at criticality the system is characterized by critical exponents. Calculation of these exponents for dimension of the system d lower than the so-called critical dimension d_c is a highly nontrivial task [2]. On the other hand for $d > d_c$ the behaviour of a given system is much simpler and critical exponents take mean-field values which are usually simple fractional numbers.

However, not everything is clearly understood above the critical dimension. One of the examples is the Ising model ($d_c = 4$) where despite intensive research serious discrepancies between analytical [3] and numerical [4] calculations still persist. Of particular interest is the value of the Binder cumulant at the critical point. Several years ago Brézin and Zinn-Justin (BJ) calculated this quantity using field theory methods [5] and only recently numerical simulations for the $d = 5$ model are able to confirm it [6]. Some other properties of the Ising model above critical dimension are still poorly explained by existing theories. For example, the theoretically predicted leading corrections to the susceptibility disagree even up to the sign with numerical simulations [4].

In addition to direct simulations of the nearest-neighbour Ising model, there are also some other ways to study the critical point of Ising model above critical dimension. For example, Luijten and Blöte used the model with $d \leq 3$ but with long-range interactions [7]. Using such an approach they confirmed with good accuracy the BJ predictions for the Binder cumulant.

In the present paper we propose yet another approach to the problem of cumulants above critical dimension. Namely, we calculate fourth- and sixth-order cumulants at the critical point of a recently introduced urn model [8]. Albeit structureless, this model exhibits a mean-field Ising-type symmetry breaking. Along an exactly located critical line, the obtained values are in a very good agreement with values predicted by BJ. Let us notice that our calculations: (i) are not affected by the inaccuracy of the location of the critical point which is a serious problem in the case of the Ising model (ii) are based on the numerical solution of the master equation which offers a much better accuracy than Monte Carlo simulations. Moreover, we calculate these cumulants at the tricritical point and show that the obtained values are also in agreement with suitably extended calculations of BJ. That both the Ising model and the (structureless) urn model have the same cumulants is a manifestation of strong universality above the upper critical dimension: at the critical point not only the lattice structure but also the lattice itself becomes irrelevant. What really matters is the type of symmetry which is broken and since in both cases it is the same Z_2 symmetry, the equality of cumulants follows.

Our urn model was motivated by recent experiments on the spatial separation of shaken sand [9]. In the present paper we are not concerned with the relation with granular matter and a more detailed justification of rules of the urn model is omitted [8]. The model is defined as follows: N particles are distributed between two urns A and B and the number of particles in each urn is denoted as M and $N - M$, respectively. Particles in a given urn (say A) are subject to thermal fluctuations and the temperature T of the urn depends on the number of particles in it as:

$$T(x) = T_0 + \Delta(1 - x), \quad (1)$$

where x is a fraction of a total number of particles in a given urn and T_0 and Δ are positive constants. (For urn A and B, $x = M/N$ and $(N - M)/N$, respectively.) Next, we define dynamics of the model [8]:

- (i) One of the N particles is selected randomly.
- (ii) With probability $\exp[\frac{-1}{T(x)}]$ the selected particle changes urns, where x is the fraction of particles in the urn of a selected particle.

To measure the difference in the occupancy of the urns we define

$$\epsilon = \frac{2M - N}{2N} = \frac{M}{N} - \frac{1}{2}. \quad (2)$$

In the steady state the flux of particles changing their positions from A to B equals to the flux from B to A. Since the selected particles are uncorrelated, the above requirement can be written as:

$$\langle M \rangle \exp\left[\frac{-1}{T(\langle M/N \rangle)}\right] = \langle N - M \rangle \exp\left[\frac{-1}{T(\langle (N - M)/N \rangle)}\right], \quad (3)$$

or equivalently

$$\left(\frac{1}{2} + \langle \epsilon \rangle\right) \exp\left[\frac{-1}{T(\frac{1}{2} + \langle \epsilon \rangle)}\right] = \left(\frac{1}{2} - \langle \epsilon \rangle\right) \exp\left[\frac{-1}{T(\frac{1}{2} - \langle \epsilon \rangle)}\right]. \quad (4)$$

Analysis of eq. (4) shows [8] that on the (Δ, T_0) phase diagram symmetric ($\epsilon = 0$) and asymmetric ($\epsilon \neq 0$) solutions are separated by the critical line which is given by the following equation

$$T_0 = \sqrt{\Delta/2} - \Delta/2, \quad 0 < \Delta < \frac{2}{3}. \quad (5)$$

The critical lines terminates at the tricritical point: $\Delta = \frac{2}{3}, T_0 = \frac{\sqrt{3}-1}{3}$. Let us notice that a random selection of particles implies basically the mean-field nature of this model. Consequently, at the critical point $\beta = 1/2$ and $\gamma \approx 1$ (measured from the divergence of the variance of the order parameter), which are ordinary mean-field exponents. However, the calculation of the dynamical exponent z gives $z = 0.50(1)$ [8] while the mean-field value is 2. We do not have convincing arguments which would explain such a small value of z . Presumably, this fact might be related with a structureless nature of our model.

Defining $p(M, t)$ as the probability that in a given urn (say A) at the time t there are M particles, the evolution of the model is described by the following master equation

$$\begin{aligned} p(M, t+1) &= \frac{N-M+1}{N} p(M-1, t) \omega(N-M+1) + \frac{M+1}{N} p(M+1, t) \omega(M+1) + \\ &\quad p(M, t) \left\{ \frac{M}{N} [1 - \omega(M)] + \frac{N-M}{N} [1 - \omega(N-M)] \right\} \text{ for } M = 1, 2, \dots, N-1 \\ p(0, t+1) &= \frac{1}{N} p(1, t) \omega(1) + p(0, t) [1 - \omega(N)], \\ p(N, t+1) &= \frac{1}{N} p(N-1, t) \omega(1) + p(N, t) [1 - \omega(N)], \end{aligned} \quad (6)$$

where $\omega(M) = \exp[\frac{-1}{T(M/N)}]$. Supplementing the above equations with initial conditions one can easily solve them numerically.

Cumulants that we calculate are defined as

$$x_4 = \frac{\langle \epsilon^4 \rangle}{\langle \epsilon^2 \rangle^2}, \quad x_6 = \frac{\langle \epsilon^6 \rangle}{\langle \epsilon^2 \rangle^3} \quad (7)$$

where

$$\langle \epsilon^n \rangle = \sum_{M=0}^N \left(\frac{M}{N} - \frac{1}{2} \right)^n p(M, \infty) \quad (8)$$

and the symbol of infinity indicates that we take the long-time (steady-state) solutions of the master equation (6). Calculations are made for $\Delta = \frac{1}{8}, \frac{1}{4}, \frac{1}{2}$ and $\frac{2}{3}$ and for each Δ the value of T_0 is calculated from eq. (5). Thus, the last point is the tricritical point and the remaining ones are critical points. Numerical results are presented in Figs. 1-4.

Before discussing our results further, let us briefly describe the BJ approach. To calculate cumulants above the critical dimension they used the Ginzburg-Landau-Wilson model. Then, they calculate the effective action restricting the expansion only to the homogeneous contributions (the lowest-mode approximation). Since at criticality the quadratic (in the order parameter) term vanishes in such an expansion and the leading term is quartic which implies that the probability distribution has the form $p(x) \sim e^{-x^4}$, where x is a rescaled order parameter. Calculations of moments for such a distribution are then elementary and one obtains

$$x_4 = \frac{1}{8\pi^2}[\Gamma(\frac{1}{4})]^4 \approx 2.188440..., \quad x_6 = \frac{3}{8\pi^2}[\Gamma(\frac{1}{4})]^4 \approx 6.565319.... \quad (9)$$

The fact that one can restrict the expansion of the free energy to the lowest order term is by no means obvious [3]. Such a restriction leads to the correct results but only above critical dimension where the model behaves according to the mean-field scenario with fluctuations playing negligible role. For $d < d_c$ additional terms in the expansion are also important and cumulants take different value. Numerical confirmation of the above results requires extensive Monte Carlo simulations, and a satisfactory confirmation was obtained only for x_4 [6,10].

Omitting detailed field theory analysis, we can extend the BJ approach to the tricritical point. At such a point also the quartic term vanishes which makes the sixth-order term the leading one and the probability distribution gets the form $p(x) \sim e^{-x^6}$. Simple calculations for such a distribution yield

$$x_4 = \frac{\Gamma(\frac{5}{6})\Gamma(\frac{1}{6})}{\Gamma(\frac{1}{2})^2} = 2, \quad x_6 = \frac{\Gamma(\frac{1}{6})^3}{6\Gamma(\frac{1}{2})^3} \approx 5.162113.... \quad (10)$$

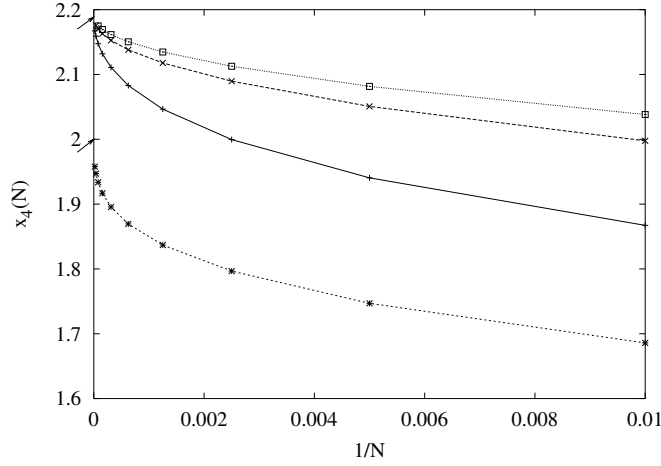


FIG. 1. The fourth-order cumulant $x_4(N)$ as a function of $1/N$ for (from top) $\Delta = 0.125, 0.25, 0.5$ and $\frac{2}{3}$ (tricritical point). Arrows indicate the BJ results for the critical and the tricritical point.

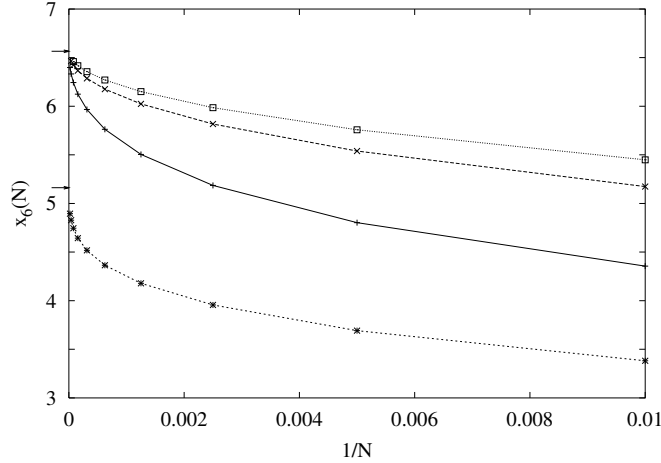


FIG. 2. The same as in Fig. 1 but for the sixth-order cumulant $x_6(N)$.

The BJ results (9)-(10) are indicated by small arrows in Figs. 1-2. Even without any extrapolation one can see, especially for critical points, a good agreement with our results. Data in Figs. 1-2 shows strong finite-size corrections. To have a better estimations of asymptotic values in the limit $N \rightarrow \infty$ we assume finite size corrections of the form

$$x_{4,6}(N) = x_{4,6}(\infty) + AN^{-\omega}. \quad (11)$$

The least-square fitting of our finite- N data to eq. (11) gives $x_{4,6}(\infty)$ which agree with BJ values (9)-(10) within the accuracy better than 0.1%. A better estimation of the correction exponent ω is obtained assuming that $x_{4,6}(\infty)$ are given by the BJ values. The exponent ω equals then the slope of the data in the logarithmic scale as presented in Figs. 3-4. Our data shows that for the critical(tricritical) point $\omega = \frac{1}{2}(\frac{1}{3})$.

Let us notice that leading finite-size corrections to the Binder cumulant in the $d = 5$ Ising model at the critical point are also of the form $N^{-0.5}$ (with N being the linear system size) [7]. Moreover, for the tricritical point but $d < d_c$ the probability distribution is known to exhibit a three-peak structure [11], which is different than the single-peak form $p(x) \sim e^{-x^6}$.

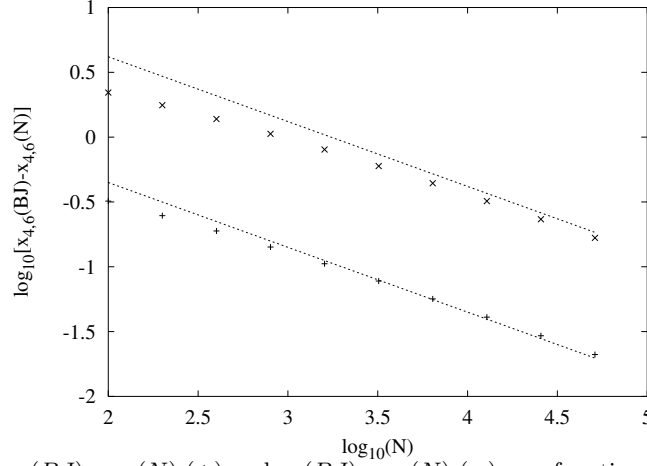


FIG. 3. Logarithmic plot of $x_4(BJ) - x_4(N)$ (+) and $x_6(BJ) - x_6(N)$ (x) as a function for N for $\Delta = 0.5$. Dotted straight lines have slope 0.5.

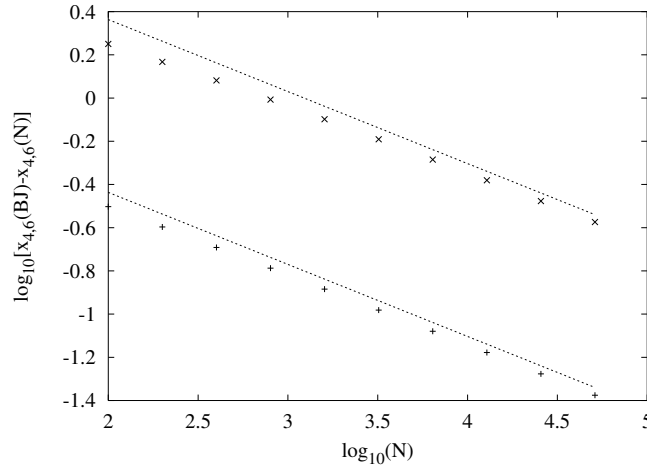


FIG. 4. Logarithmic plot of $x_4(BJ) - x_4(N)$ (+) and $x_6(BJ) - x_6(N)$ (x) as a function for N for $\Delta = \frac{2}{3}$ (tricritical point). Dotted straight lines have slope $\frac{1}{3}$.

In summary, we calculated fourth- and sixth-order cumulants at the critical and tricritical points in an urn model which undergoes a symmetry breaking transition. Our results confirm that, as predicted by Brézin and Zinn-Justin, the critical probability distributions of the rescaled order parameter has the form $p(x) \sim e^{-x^4}$. Similarly, for the tricritical point our results suggest that $p(x) \sim e^{-x^6}$.

Although in our opinion convincing, the results are obtained using numerical methods. It would be desirable to have analytical arguments for the generation of such probability distributions. It seems that for the presented urn model this might be easier than for the Ising-type models. Let us notice that for the simplest urn model, which was introduced by Ehrenfest [12], the steady-state probability distribution can be calculated exactly in the continuum limit of the master equation and the result has the form $p(x) \sim e^{-x^2}$, where x is now proportional to the difference of occupancy ϵ . In the Ehrenfest model there is no critical point and we expect that a distribution of the type e^{-x^2}

might characterize our model but off the critical line (in the symmetric phase). We hope that when suitably extended, an analytic approach to our model might extract critical and tricritical distributions as well. Such an approach is left as a future problem.

ACKNOWLEDGMENTS

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